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We study different characterizations of the pointwise Hölder spaces $C^s(x_0)$, including rate of approximation by smooth functions and iterated differences. As an application of our results we study the class of functions that are Hölder exponents and prove that the Hölder exponent of a continuous function is the limit inferior of a sequence of continuous functions, thereby improving a theorem of S. Jaffard.

1. INTRODUCTION

A function f belongs to the pointwise Hölder space $C^s(x_0)$, where $s > 0$, if there is a polynomial P of degree less than s such that $|f(x) - P(x - x_0)| \leqslant C|x - x_0|^s$ in a neighborhood of x_0. In this paper we are interested in finding other ways of characterizing these spaces.

It is well known that the regularity, both local and global, of a function f is reflected by the decay of its wavelet coefficients. For example, if $\{2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$ is an orthogonal wavelet basis of $L^2(\mathbb{R})$ as constructed in [9], with ψ belonging to the Schwartz class $\mathcal{S}(\mathbb{R})$ of rapidly decaying smooth functions, and we let $\alpha_{j,k} = \langle f, 2^j\psi(2^j x - k)\rangle$ denote the wavelet coefficients of f, then f belongs to the global Hölder space $C^s(\mathbb{R})$ if and only if f is bounded and $|\alpha_{j,k}| \leqslant C2^{-js}$. The same holds in the n-dimensional case except that one needs several wavelets; see [9].

As for the pointwise Hölder spaces, $f \in C^s(x_0)$ implies that

$$|\alpha_{j,k}| \leqslant C(2^{-j} + |x_0 - k2^{-j}|)^s, \tag{1.1}$$

but the converse does not hold. To obtain a converse we have to assume that, in addition to (1.1), f satisfies a global regularity condition, and we can only conclude that $|f(x) - P(x - x_0)| \leqslant C|x_0 - x_0|^s \log(1/|x - x_0|)$. See [4] for proofs, counterexamples, and more details.

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In Section 2 we discuss some alternative approaches that give exact characterizations of $C^s(x_0)$, mainly iterated differences and rate of approximation by smooth functions, with regularization by band-limited functions as a concrete example. We will also see that the definition of $C^s(x_0)$ should be modified when s is an integer. In Section 3 we consider instead the rate of approximation by a multiresolution analysis. As an application of our results we then study Hölder exponents in Section 4, using the characterizations from Section 2.

The Hölder exponent of a locally bounded function f at a point x_0, denoted by $\alpha(x_0)$, is defined as the supremum of all s such that $f \in C^s(x_0)$. Thus α is a function taking its values in $[0, \infty]$ which measures the pointwise regularity of f at each point.

The Hölder exponent $\alpha(x)$ may be a very irregular and complicated function, and a more tractable description of the regularity of f can be obtained by considering only the spectrum of singularities, which is defined as the (Hausdorff) dimensions of the level sets of $\alpha(x)$. This definition is obviously very unstable from a numerical point of view, and the so called multifractal formalism was introduced to give numerically stable algorithms to compute the spectrum of singularities. For more details on the multifractal formalism and a discussion of its validity, see [6].

The question of which functions may be Hölder exponents was first raised by Lévy-Véhel and partially answered in [1]. Jaffard [7] then proved that a non-negative function $\alpha(x)$ is the Hölder exponent of some function f satisfying a (rather weak) global regularity condition if and only if $\alpha(x)$ can be written as a limit inferior of a sequence of continuous functions. The global regularity assumption comes from the fact that wavelets are used in both the analysis and the construction part. By using instead our characterizations of $C^s(x_0)$ in the analysis part we can drop the global regularity condition in Jaffard's theorem and obtain the following result, which we prove in Section 4.

THEOREM 1. *Let α be the pointwise Hölder exponent of a continuous function f. Then α can be written as a limit inferior of a sequence of continuous functions. Conversely, for each α of this type there is a continuous function having α as its Hölder exponent.*

2. CHARACTERIZATION OF $C^s(x_0)$

Let us first recall some facts about global Hölder spaces $C^s(\mathbb{R}^n)$, where we assume that $s > 0$. Let m be the largest integer not exceeding s. $C^s(\mathbb{R}^n)$ is then the set of bounded, m times continuously differentiable functions f, with all the partial derivatives $\partial^\alpha f$ of order m satisfying

$$|\partial^\alpha f(x) - \partial^\alpha f(y)| \leqslant C|x-y|^{s-m} \tag{2.1}$$

for all $x, y \in \mathbb{R}^n$. Note that since we assume f to be bounded, this condition is automatically satisfied for large $|x-y|$, and we may restrict attention to, say, $|x-y| \leqslant 1$. The condition (2.1) is equivalent to requiring that

$$|f(x) - P_{x_0}(x-x_0)| \leqslant C|x-x_0|^s,$$

uniformly in x and x_0, where P_{x_0} denotes the Taylor polynomial of f at x_0.

There are several other equivalent ways of characterizing global Hölder spaces. One of them is in terms of iterated differences. For any $h \in \mathbb{R}^n$, we let Δ_h denote the (forward) difference operator, defined by

$$\Delta_h f(x) = f(x+h) - f(x),$$

and $\Delta_h^N = \Delta_h \cdots \Delta_h$ its Nth iterate. Then, at least for non-integer s, we have $f \in C^s(\mathbb{R}^n)$ if and only if $f \in L^\infty(\mathbb{R}^n)$ and, with N denoting an arbitrary integer larger than s,

$$\|\Delta_h^N f\|_{L^\infty} \leqslant C|h|^s$$

for all $h \in \mathbb{R}^n$. For integer s this condition does not characterize the set of s times continuously differentiable functions, or even the set of functions with Lipschitz continuous $(s-1)$th derivative, but a somewhat larger space, which turns out to be the more natural one, and one usually adopts the convention that $C^s(\mathbb{R}^n)$ denotes this space. For $s = 1$ this space is known as the Zygmund class of smooth functions Λ_*, and a typical example of a function in Λ_* which is not Lipschitz continuous is $f(x) = x_1 \log|x|$. More generally, $f(x) = x_1^m \log|x|$ belongs to the Zygmund-type $C^m(\mathbb{R}^n)$ space, but we do not have $f(x) = P(x) + O(|x|^m)$.

Another characterization of $C^s(\mathbb{R}^n)$ is in terms of the rate of approximation by smooth functions. With the above convention for integer s, a function f belongs to $C^s(\mathbb{R}^n)$ if and only if it can be written

$$f = \sum_{j=0}^{\infty} g_j,$$

with

$$\|\partial^\alpha g_j\|_{L^\infty} \leqslant C 2^{j(|\alpha|-s)} \tag{2.2}$$

for $|\alpha| \leqslant m$, where m is the smallest integer (strictly) greater than s [12]. With $f_j = \sum_{k=0}^{j} g_k$, it follows that

$$\|f - f_j\|_{L^\infty} \leqslant C 2^{-js}, \tag{2.3}$$

and the condition for $f \in C^s(\mathbb{R}^n)$ can equivalently be formulated as the existence of a sequence of $C^m(\mathbb{R}^n)$ functions f_j satisfying (2.3) and such that (2.2) holds with $g_j = f_j - f_{j-1}$.

A related characterization is by Littlewood–Paley theory. Here one starts by choosing a function φ in the Schwartz class $\mathcal{S}$, whose Fourier transform $\hat{\varphi}(\xi)$ vanishes for $|\xi| \geqslant 2$ and is identically 1 for $|\xi| \leqslant 1$. One then defines the ‘‘partial sum’’ operators $S_j f = f * \varphi_{2^{-j}}$, where we use the notation $\varphi_t(x) = t^{-n}\varphi(x/t)$, and the difference operators

$$\Delta_j f = S_j f - S_{j-1} f = f * \psi_{2^{-j}},$$

where $\psi = \varphi - \varphi_2 \in \mathcal{S}$. $\hat{\psi}_{2^{-j}}$ will then be supported in $2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}$. With this notation, $f \in C^s(\mathbb{R}^n)$ if and only if $f \in L^\infty$ and

$$\|\Delta_j f\|_{L^\infty} \leqslant C2^{-js}.$$

A reference for the above facts and more general ones is [11].

For the pointwise Hölder spaces $C^s(x_0)$, the situation is somewhat complicated by the fact that we only consider regularity at one point x_0. Note that even if $f \in C^s(x_0)$ with s large, f may not be differentiable except at x_0, and higher derivatives may not exist anywhere. However, we still have characterization along the same lines as for the global Hölder spaces, as described by the following theorem.

THEOREM 2. *Assume that $s > 0$, and let m be the smallest integer strictly greater than s, and N any integer strictly greater than s. Let f be a locally integrable function defined in a neighborhood U of a point $x_0 \in \mathbb{R}^n$. Then the following properties of f are equivalent.*

(i) *We have*

$$|\Delta_h^N f(x)| \leqslant C(|h| + |x - x_0|)^s \tag{2.4}$$

for all sufficiently small $|h| + |x - x_0|$.

(ii) *There is a neighborhood U of x_0 and a sequence of functions $f_j \in C^m(U)$ such that*

$$|f(x) - f_j(x)| \leqslant C(2^{-j} + |x - x_0|)^s \tag{2.5}$$

for all $x \in U$, and, with $g_j = f_j - f_{j-1}$,

$$|\partial^\alpha g_j(x)| \leqslant C2^{j|\alpha|}(2^{-j} + |x - x_0|)^s \tag{2.6}$$

for any multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant m$.

(iii) *With $\tilde{f}$ defined on $\mathbb{R}^n$ by $\tilde{f} = f\chi$, where $\chi \in C_0^\infty(\mathbb{R}^n)$ is identically 1 in a neighborhood of x_0, we have*

$$|\tilde{f}(x) - S_j\tilde{f}(x)| \leqslant C(2^{-j} + |x - x_0|)^s. \tag{2.7}$$

For non-integer s, these properties are also equivalent to the following one.

(iv) *There is a polynomial P of degree less than s such that*

$$|f(x) - P(x - x_0)| \leqslant C|x - x_0|^s \tag{2.8}$$

in a neighborhood of x_0.

In view of this theorem, it is natural to choose to define $C^s(x_0)$ for integer s by (i)–(iii) instead of (iv). As long as one is only interested in the Hölder exponent $\alpha(x_0)$ this of course makes no difference.

Remark. In (iii) the purpose of introducing the function $\tilde{f}$ is simply to make $S_j f = f * \varphi_{2^{-j}}$ well defined. If f is defined on the whole of $\mathbb{R}^n$ with a growth condition at infinity, say that f grows at most polynomially, then one can do without this step. Another way to get around this inconvenience is to define a Littlewood–Paley analysis with ϕ and ψ compactly supported, cf. [11], but we will not discuss this further.

Note that, unlike in the $C^s(\mathbb{R}^n)$ case, an assumption of the type (2.5) or (2.7) is needed in (ii) and (iii); only (2.6) combined with, e.g., $f = \sum_{j=0}^{\infty} g_j$ in the sense of distributions does not suffice. In the special case where g_j is a Littlewood–Paley block, as in (iii), the latter condition characterizes instead exactly the 2-microlocal space $C_{x_0}^{s,-s}$, which is strictly larger than $C^s(x_0)$ and contains true distributions. The condition $f \in C_{x_0}^{s,-s}$ combined with some global regularity assumption on f, say $f \in C^\sigma(U)$ for some $\sigma > 0$, allows us to deduce that $f \in C^{s-\epsilon}(x_0)$ for any $\epsilon > 0$, or even $|f(x) - P(x - x_0)| \leqslant |x - x_0|^s \log(1/|x - x_0|)$, but does not imply $f \in C^s(x_0)$. For a discussion of this and 2-microlocal spaces in general, see [5] or [8].

Proof of Theorem 2. We will first prove that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii), and then (iv) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iv). To prove that (iii) $\Rightarrow$ (ii) we only have to take $f_j = S_j \tilde{f}$. Then clearly (2.5) holds in the neighborhood of x_0 where $\chi = 1$. It is also clear that (2.6) holds (globally) in the case $\alpha = 0$, and for general α it follows from the weighted Bernstein inequality in the following lemma.

LEMMA 1. *For any $s \geqslant 0$ there is a constant C such that*

$$\left\| \frac{\partial^\alpha f(x)}{(h + |x - x_0|)^s} \right\|_{L^p(\mathbb{R}^n)} \leqslant \left(\frac{C}{h} \right)^{|\alpha|} \left\| \frac{f(x)}{(h + |x - x_0|)^s} \right\|_{L^p(\mathbb{R}^n)}$$

for all tempered distributions f with supp $\hat{f} \subset \{\xi : |\xi| \leqslant 1/h\}$ and all $p \in [1, \infty]$.

Proof. Without loss of generality we may assume that $x_0 = 0$. Let θ denote a function in the Schwartz class $\mathcal{S}$ whose Fourier transform is compactly supported and identically 1 on the unit ball. Then $\hat{f}(\xi) = \hat{f}(\xi)\hat{\theta}(h\xi)$, so that $f = f * \theta_h$ and $\partial^\alpha f = h^{-|\alpha|} f * (\partial^\alpha \theta)_h$. Hence we have

$$\begin{aligned} \left| \frac{\partial^\alpha f(x)}{(h + |x|)^s} \right| &\leqslant h^{-|\alpha|} \int \frac{|f(x - y)|}{(h + |x|)^s} (\partial^\alpha \theta)_h(y)\, dy \\ &\leqslant h^{-|\alpha|} \int \frac{|f(x - y)|}{(h + |x - y|)^s} \left(1 + \left| \frac{y}{h} \right| \right)^s (\partial^\alpha \theta)_h(y)\, dy, \end{aligned}$$

where we have used the elementary inequality $1 + |x - y| \leqslant (1 + |x|)(1 + |y|)$. Young's inequality followed by a change of variables finally yields

$$\begin{aligned} \left\| \frac{\partial^\alpha f(x)}{(h + |x|)^s} \right\|_{L^p} &\leqslant h^{-|\alpha|} \| (1 + |y|)^s \partial^\alpha \theta(y) \|_{L^1} \left\| \frac{f(x)}{(h + |x|)^s} \right\|_{L^p} \\ &= \frac{C_{s,\alpha}}{h^{|\alpha|}} \left\| \frac{f(x)}{(h + |x|)^s} \right\|_{L^p}, \end{aligned}$$

which is the desired inequality, at least with the constant depending on α. To prove the lemma exactly as stated, we simply use induction on $|\alpha|$. ■

We next turn to the implication (ii) ⇒ (i), and start by writing

$$|\Delta_h^N f(x)| \leqslant |\Delta_h^N (f - f_j)(x)| + |\Delta_h^N f_j(x)|. \tag{2.9}$$

For the first term (2.5) immediately gives

$$|\Delta_h^N (f - f_j)(x)| \leqslant C(2^{-j} + N|h| + |x - x_0|)^s \leqslant C(|h| + |x - x_0|)^s,$$

if we, given x and h, choose j so that $2^{-j-1} < N|h| + |x - x_0| \leqslant 2^{-j}$. For the second term, we note that

$$|\Delta_h^N f_j(x)| \leqslant C(N|h| + |x - x_0|)^m \|f_j\|_{\dot{C}^m(B_j)},$$

where $B_j = \{x : |x - x_0| \leqslant 2^{-j}\}$ and

$$\|f\|_{\dot{C}^m(B)} = \sup_{x \in B} \sum_{|\alpha| = m} |\partial^\alpha f(x)|.$$

Since $f_j = f_{j_0} + g_{j_0+1} + \cdots + g_j$ and, by (2.6), $\|g_k\|_{\dot{C}^m(B_k)} \leqslant C 2^{k(m-s)}$, we obtain

$$\begin{aligned} \|f_j\|_{\dot{C}^m(B_j)} &\leqslant \|f_{j_0}\|_{\dot{C}^m(B_j)} + \sum_{k=j_0+1}^{j} \|g_k\|_{\dot{C}^m(B_k)} \\ &\leqslant C_0 + C \sum_{k=j_0+1}^{j} 2^{k(m-s)} \leqslant C 2^{j(m-s)} \end{aligned} \tag{2.10}$$

for $j \geqslant j_0$, where j_0 is some fixed integer with $B_{j_0} \subset U$. This means that the second term in (2.9) is bounded by $C(|h| + |x - x_0|)^m 2^{j(m-s)} \leqslant C(|h| + |x - x_0|)^s$, and (2.4) follows.

We now prove that (i) ⇒ (iii). First note that, after replacing f by $f\chi$, we can assume that f is defined on the whole of $\mathbb{R}^n$ and satisfies (2.4) for all x and h. We then write

$$\Delta_h^N f(x) = \sum_{k=0}^{N} \binom{N}{k} (-1)^{N-k} f(x + kh), \tag{2.11}$$

and average this expression over different h by multiplying with $\varphi_{2^{-j}}(-h)$, with φ as in the definition of a Littlewood–Paley decomposition above, and then integrating with respect to h. A change of variables shows that

$$\int f(x + kh)\varphi_{2^{-j}}(-h)\,dh = f * \varphi_{k2^{-j}}(x),$$

so we end up with

$$\left|\sum_{k=0}^{N}\binom{N}{k}(-1)^{N-k} f * \varphi_{k2^{-j}}(x)\right| \leqslant \int |\Delta_h^N f(x)\varphi_{2^{-j}}(-h)|\, dh$$

$$\leqslant C\int (|h| + |x - x_0|)^s |\varphi_{2^{-j}}(h)|\, dh \leqslant C(2^{-j} + |x - x_0|)^s, \quad (2.12)$$

where the last inequality follows from the fact that $\int |h|^s |\varphi_r(h)|\, dh = cr^s$, where $c = \int |h|^s |\varphi(h)|\, dh$ is finite due to the rapid decay of φ. Now the left-hand side of (2.12) can be written

$$|f(x) - S_j f(x) + \sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k f * \psi^k_{2^{-j}}(x)|, \quad (2.13)$$

with $\psi^k = \varphi_k - \varphi_{k+1}$, and (2.7) follows if we note that $\hat{\psi}^k$ are supported in annuli $1/(k+1) \leqslant |\xi| \leqslant 2/k$, and use the following lemma to estimate $f * \psi^k_{2^{-j}}$.

LEMMA 2. *Assume that, for some positive integer N,*

$$|\Delta_h^N f(x)| \leqslant (|h| + |x - x_0|)^s \quad (2.14)$$

for all $x, h \in \mathbb{R}^n$. If ψ is a function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ whose Fourier transform is supported in $a \leqslant |\xi| \leqslant b$ for some $a, b > 0$, then

$$|f * \psi_h(x)| \leqslant C(|h| + |x - x_0|)^s. \quad (2.15)$$

Remark. Another way of phrasing the conclusion in the lemma is that f belongs to the 2-microlocal space $C^{s,-s}_{x_0}$.

Proof of Lemma 2. We first decompose ψ in such a way that we can extract a factor of the type Δ_h^N from each term. To do this, we work on the Fourier side, and define a partition of unity as follows. Let $\eta(t)$ be a non-negative $C^\infty(\mathbb{R})$ function which vanishes for $|t| \leqslant 1/2\sqrt{n}$ and is identically 1 for $|t| \geqslant 1/\sqrt{n}$. Then the functions $\theta_k(\xi) = \eta(\xi_k)/\sum_{j=1}^n \eta(\xi_j)$, $k = 1, \ldots, n$, define a C^∞ partition of unity in a neighborhood of the support of $\hat{\psi}$, and we have

$$\hat{\psi}(\xi) = \sum_{k=1}^{n} \theta_k(\xi)\hat{\psi}(\xi) = \sum_{k=1}^{n} (e^{ic\xi_k} - 1)^N \hat{\psi}^k(\xi),$$

with $\hat{\psi}^k(\xi) = \theta_k(\xi)\hat{\psi}(\xi)/(e^{ic\xi_k} - 1)^N$. By choosing $c = 1/2b$ we ensure that the factor $(e^{ic\xi_k} - 1)$ never vanishes on the support of $\theta_k\hat{\psi}$, and consequently we have $\hat{\psi}^k \in \mathcal{S}$. Taking the inverse Fourier transform we get

$$\psi = \sum_{k=1}^{n} \Delta^N_{ce_k} * \psi^k,$$

where $e_1, \ldots, e_n$ are the canonical basis vectors in $\mathbb{R}^n$, and hence

$$f * \psi_h = \sum_{k=1}^{n} f * (\Delta^N_{che_k} \psi^k_h) = \sum_{k=1}^{n} (\Delta^N_{che_k} f) * \psi^k_h.$$

Combining this with (2.14) and the rapid decay of the ψ^k we obtain (2.15). ■

Returning to the proof of Theorem 2, we see that the implication (iv) ⇒ (i) follows immediately if we note that $\Delta^N_h P(x) = 0$, and we finally prove that (ii) ⇒ (iv) by adapting an argument from [10].

Let $P_j(x) = \sum_{|\alpha|<s} \partial^\alpha f_j(x_0) x^\alpha/\alpha!$ be the Taylor expansion of f_j at x_0. From the assumption (2.6) it follows that the coefficients of P_j converge. To see this, we note that

$$|\partial^\alpha f_j(x_0) - \partial^\alpha f_{j-1}(x_0)| \leqslant C 2^{j(|\alpha|-s)},$$

where the right-hand side is summable as $j \to \infty$ when $|\alpha| < s$. Hence $\partial^\alpha f_j(x_0)$ converges to a limit λ_α, and summing a telescoping series we also see that

$$|\partial^\alpha f_j(x_0) - \lambda_\alpha| \leqslant C 2^{j(|\alpha|-s)}.$$

Now we define $P(x) = \sum_{|\alpha|<s} \lambda_\alpha x^\alpha/\alpha!$ and write

$$\begin{aligned} |f(x) - P(x-x_0)| \leqslant |f(x) - f_j(x)| &+ |f_j(x) - P_j(x-x_0)| \\ &+ |P_j(x-x_0) - P(x-x_0)|. \end{aligned} \tag{2.16}$$

Given $x \neq x_0$ we choose $j \in \mathbb{Z}$ so that $2^{-j-1} < |x - x_0| \leqslant 2^{-j}$. Then, by (2.5),

$$|f(x) - f_j(x)| \leqslant C(2^{-j} + |x-x_0|)^s \leqslant C|x-x_0|^s,$$

and, by obvious estimates,

$$\begin{aligned} |P_j(x-x_0) - P(x-x_0)| &\leqslant \sum_{|\alpha|<s} |\partial^\alpha f_j(x_0) - \lambda_\alpha| |x-x_0|^{|\alpha|}/\alpha! \\ &\leqslant C \sum_{|\alpha|<s} 2^{j(|\alpha|-s)} |x-x_0|^{|\alpha|} \leqslant C|x-x_0|^s. \end{aligned}$$

For the remaining term in (2.16), Taylor's formula and the estimate in (2.10) give

$$\begin{aligned} |f_j(x) - P_j(x-x_0)| &\leqslant C|x-x_0|^m \|f_j\|_{\dot{C}^m(B_j)} \\ &\leqslant C|x-x_0|^m 2^{j(m-s)} \leqslant C|x-x_0|^s \end{aligned}$$

for small enough x, so we have $|f(x) - P(x - x_0)| \leqslant C|x - x_0|^s$, which concludes the proof of Theorem 2. ■

3. CHARACTERIZATION USING A MULTIRESOLUTION ANALYSIS

Another way of obtaining approximating sequences f_j is to approximate f by its projections on a ladder of multiresolution spaces. Let us first briefly recall the definition and some properties of multiresolution analysis.

DEFINITION. A multiresolution analysis of $L^2(\mathbb{R}^n)$ is a sequence $(V_j)_{j\in\mathbb{Z}}$ of closed linear subspaces of $L^2(\mathbb{R}^n)$ such that

- $V_j \subset V_{j+1}$, $\cap V_j = \{0\}$, and $\cup V_j$ is dense in $L^2(\mathbb{R}^n)$.
- $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$.
- V_0 has an orthonormal basis $\{\varphi(x - k)\}_{k\in\mathbb{Z}^n}$.

The function φ is called scaling function.

For our purposes we also need to impose some regularity on φ. Let $\mathcal{S}_r(\mathbb{R}^n)$, where $r \in \mathbb{N}$, be the set of functions f such that $|\partial^\alpha f| \leqslant C_m(1 + |x|)^{-m}$ for $|\alpha| \leqslant r$ and $m \in \mathbb{N}$. A multiresolution analysis is said to be r-regular if the scaling function can be chosen in $\mathcal{S}_r(\mathbb{R}^n)$.

We denote by P_j the orthogonal projection operator onto V_j, i.e.,

$$P_j f(x) = \int K_j(x, y) f(y) dy,$$

where

$$K(x, y) = \sum_{k\in\mathbb{Z}^n} \varphi(x - k)\overline{\varphi(y - k)}$$

is the kernel distribution of P_0, and $K_j(x, y) = 2^{jn} K(2^j x, 2^j y)$. From $\varphi \in \mathcal{S}_r(\mathbb{R}^n)$ it follows that

$$|\partial_x^\alpha \partial_y^\beta K(x, y)| \leqslant C_m(1 + |x - y|)^{-m} \tag{3.1}$$

for $|\alpha|$, $|\beta| \leqslant r$ and any $m \in \mathbb{N}$.

Note that $P_j f$ is well defined for any f that is polynomially bounded, i.e., that satisfies $|f(x)| \leqslant C(1 + |x|)^m$ for some $m \in \mathbb{N}$. In what follows V_j will denote the set of sums $\Sigma_{k\in\mathbb{Z}^n} a_k \varphi(2^j x - k)$ with $\{a_k\}$ polynomially bounded, rather than a subspace of $L^2(\mathbb{R}^n)$.

An important fact is that for an r-regular multiresolution analysis we have $P_j p = p$ for all polynomials p of total degree at most r; see [2] or [9]. Without too much effort, one can prove that the integer translates of a function φ generate all polynomials of degree up to r if and only if $\hat{\varphi}(0) \neq 0$ and $\partial^\alpha \hat{\varphi}(2\pi k) = 0$ for $|\alpha| \leqslant r$ and $k \in \mathbb{Z}^n \setminus \{0\}$. This is known as the Strang–Fix conditions [3] and applies in particular to scaling functions φ.

We can now formulate the following theorem.

THEOREM 3. *Let $(V_j)_{j\in\mathbb{Z}}$ be an r-regular multiresolution analysis, and $s \in (0, r)$. Let f be a polynomially bounded locally integrable function on $\mathbb{R}^n$. Then $f \in C^s(x_0)$ if and only if*

$$|f(x) - P_j f(x)| \leqslant C(2^{-j} + |x - x_0|)^s \tag{3.2}$$

for all sufficiently small $2^{-j} + |x - x_0|$.

Proof. The sufficiency of (3.2) follows from the weighted Bernstein inequality in Lemma 3 below in exactly the same way as the implication (iii) $\Rightarrow$ (ii) in Theorem 2 follows from Lemma 1, with $f_j = P_j f$.

LEMMA 3. *Let $(V_j)_{j\in\mathbb{Z}}$ be an r-regular multiresolution analysis. For any $s \geqslant 0$ and any multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant r$ there is a constant C such that*

$$\left\|\frac{\partial^\alpha f(x)}{(2^{-j} + |x - x_0|)^s}\right\|_{L^p(\mathbb{R}^n)} \leqslant C2^{j|\alpha|}\left\|\frac{f(x)}{(2^{-j} + |x - x_0|)^s}\right\|_{L^p(\mathbb{R}^n)}$$

for all $f \in V_j$ and all $p \in [1, \infty]$.

This lemma is proved in exactly the same way as Lemma 1, using the fact that $\partial^\alpha P_j f = 2^{j|\alpha|} \int (\partial_x^\alpha K)_j(x, y) f(y)dy$, and the estimate (3.1) for $\partial_x^\alpha K$.

The necessity of (3.2) is almost obvious when s is not an integer. Then we only have to write $|f - P_j f| = |f - p - P_j(f - p)| \leqslant |f - p| + |P_j(f - p)|$, with p a polynomial as in property (iv) of Theorem 2, and then estimate $|f - p|$ by (2.8) and use the decay estimate (3.1) for the kernel K.

For a proof that applies for arbitrary $s \in (0, r)$, we can mimic the proof of the implication (i) $\Rightarrow$ (iii) in Theorem 2, only this time we multiply (2.11) by $K_j(x, x + h)$ and integrate. The right-hand side is estimated in exactly the same way, using this time the bound (3.1) for $|K(x, y)|$. Changing variables in the integrals in the left-hand side leads to

$$\begin{aligned}\int f(x + kh)K_j(x, x + h)dh &= \int f(x - y)\frac{2^j}{k}K\left(2^j x, 2^j x - \frac{2^j}{k}y\right)dy\\ &= f * \theta_{k2^{-j}}(x),\end{aligned}$$

with $\theta(y) = K(\tilde{x}, \tilde{x} - y)$ and $\tilde{x} = 2^j x$, so we arrive at

$$|f(x) - P_j f(x) + \sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k f * w^k_{2^{-j}}(x)| \leqslant C(2^{-j} + |x - x_0|)^s,$$

with $w^k = \theta_k - \theta_{k+1}$, and the desired estimate for $|f - P_j f|$ follows if we show that $|f * w_h^k(x)| \leqslant C(|h| + |x - x_0|)^s$. To do this we apply the following lemma, which is a discrete version of a lemma in [9], to each of the w^k, with $s = r$.

LEMMA 4. *Assume that* $f \in \mathcal{S}_r(\mathbb{R}^n)$ *and* $\sum_{k\in\mathbb{Z}^n} k^\alpha f(x-k) = 0$ *for* $|\alpha| \leqslant p$, *where* $p \in \mathbb{N}$. *Then we have* $f = \sum_{|\alpha|=p} \Delta^\alpha f_\alpha$, *with* $f_\alpha \in \mathcal{S}_r(\mathbb{R}^n)$. *If f depends on an implicit parameter, with decay estimates uniform in this parameter, then the same holds for each* f_α.

Here we use the notation $\Delta^\alpha = \Delta_{e_1}^{\alpha_1} \cdots \Delta_{e_n}^{\alpha_n}$. It will be useful to note that the condition in the lemma can equivalently be formulated in the Fourier domain as

$$\partial^\alpha \hat{f}(2\pi m) = 0, \qquad |\alpha| \leqslant p,\ m \in \mathbb{Z}^n. \tag{3.3}$$

This follows from the Poisson summation formula, in the same way as the Strang–Fix conditions. We omit the details.

To check that w^k satisfy (3.3), we first recall that $\theta(y) = K(\tilde{x}, \tilde{x} - y) = \sum_{k\in\mathbb{Z}^n} \varphi(\tilde{x} - k)\overline{\varphi(\tilde{x} - k - y)}$, which implies

$$\hat{\theta}(\xi) = \sum_{k\in\mathbb{Z}^n} \varphi(\tilde{x} - k) e^{-i\xi(\tilde{x}-k)} \overline{\hat{\varphi}(\xi)}.$$

Since $\partial^\alpha \hat{\varphi}(2\pi m) = 0$ for $m \neq 0$ by the Strang–Fix conditions, it follows that $\partial^\alpha \hat{\theta}(2\pi m) = 0$, and hence $\partial^\alpha \hat{w}^k(2\pi m) = 0$ for $|\alpha| \leqslant r$ and $m \neq 0$. For $m = 0$ we have, for $|\alpha| \leqslant r$,

$$\partial^\alpha \hat{\theta}(0) = \int K(\tilde{x}, \tilde{x} - y)(-iy)^\alpha dy = i^{-|\alpha|} \int K(\tilde{x}, y)(\tilde{x} - y)^\alpha dy = \delta_{\alpha,0},$$

since $P_j p = p$ for polynomials of degree $\leqslant r$. Hence $\partial^\alpha \hat{w}^k(0) = 0$.

Lemma 4 thus implies that

$$f * w_h^k = \sum_{|\alpha|=r} f * (\Delta^\alpha w_\alpha^k)_h = \sum (w_\alpha^k)_h * \Delta_h^\alpha f,$$

with $w_\alpha^k \in \mathcal{S}_r(\mathbb{R}^n)$. The differences $\Delta_h^\alpha f$ are not exactly of the same type as in Theorem 2, but one easily verifies from (ii) in Theorem 2 that the same estimates hold for general iterated differences $\Delta_{h_1} \cdots \Delta_{h_N} f$, with $|h| = \max(|h_1|, \dots, |h_N|)$. Applying this to $\Delta_h^\alpha f$ and using the rapid decay of w_α^k, we obtain the desired estimate for $f * w_h^k$, which concludes the proof of Theorem 3. ■

Proof of Lemma 4. We prove the lemma by induction on the dimension n, starting with the case $n = 1$. If we define $f_1(x) = \sum_{k=-\infty}^{-1} f(x + k)$, then $f = \Delta_1 f_1$. Since $f \in \mathcal{S}_r(\mathbb{R})$, we also have $f_1 \in \mathcal{S}_r(\mathbb{R})$, provided that $\lim_{x\to\infty} f(x) = 0$. But this limit exists since $f \in \mathcal{S}_r$, and is 0 due to the assumption $\sum_{k\in\mathbb{Z}} f(x - k) = 0$. Furthermore, partial summation, taking advantage of the rapid decay of f_1, shows that $\sum_{k\in\mathbb{Z}} k^j f_1(x - k) = 0$ for $j = 0, 1, \dots, p - 1$. Repeating this argument s times, we obtain $f = \Delta^p f_p$, with $f_p \in \mathcal{S}_r(\mathbb{R})$.

For the induction step we write $x = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$, and form

$$g_j(x', x_n) = \sum_{k_n\in\mathbb{Z}} (x_n - k_n)^j f(x', x_n - k_n), \qquad j = 0, 1, \dots, p.$$

For each fixed x_n, g_j are functions in $\mathcal{S}_r(\mathbb{R}^{n-1})$ with $\sum_{k' \in \mathbb{Z}^{n-1}} (x' - k')^\beta g_j(x' - k') = 0$ for $|\beta| \leqslant p - j$, so by the induction hypothesis we have $g_j = \sum_{|\beta|=p-j} \Delta^\beta g_{\beta,j}$, with $g_{\beta,j} \in \mathcal{S}_r(\mathbb{R}^{n-1})$.

Now form

$$r(x', x_n) = f(x', x_n) - \sum_{j=0}^{p} g_j(x', x_n)\varphi_j(x_n),$$

where $\varphi_j \in \mathcal{S}(\mathbb{R})$ are chosen so that $\sum_{k \in \mathbb{Z}} (t-k)^l \varphi_j(t-k) = \delta_{j,l}$. For fixed x', one then immediately verifies that $\sum_{k \in \mathbb{Z}} (t-k)^l r(t-k) = 0$ for $l = 0, 1, \ldots, p$, so the $n = 1$ case of the lemma implies that $r = \Delta^p h$, with $h \in \mathcal{S}_r(\mathbb{R})$. Summing up, we have

$$f(x) = \sum_{j=0}^{p} \sum_{|\beta|=p-j} \Delta^{(\beta,j)} g_{\beta,j}(x', x_n)\varphi_j(x_n) + \Delta^{(0,p)} h(x', x_n),$$

which is a decomposition of the desired form. From the construction it is also clear that $g_{\beta,j}(x)\varphi_j(x_n)$ and $h(x)$, regarded as functions on $\mathbb{R}^n$, indeed do belong to $\mathcal{S}_r(\mathbb{R}^n)$. ■

Remark. For the functions f_α in Lemma 4, we also have $\sum_{k \in \mathbb{Z}^n} f_\alpha(x - k) = 0$, though we will not make use of this fact.

4. CHARACTERIZATION OF HÖLDER EXPONENTS

In this section we use our characterizations of pointwise Hölder spaces to prove Theorem 1, which is stated in the introduction.

Proof of the direct part of Theorem 1. By a partition of unity we may assume that f is compactly supported and hence bounded. The Hölder exponent $\alpha(x_0)$ can be characterized with any of the four conditions in Theorem 2 or the one in Theorem 3. We choose (iii) in Theorem 2, which implies that $\alpha(x_0)$ is the largest s such that, for arbitrary $\epsilon > 0$,

$$|f(x) - f * \varphi_h(x)| \leqslant C_\epsilon(|h| + |x - x_0|)^{s-\epsilon} \qquad \text{for } |h| + |x - x_0| < \eta(\epsilon).$$

Taking the logarithm and simplifying, considering only $|h| + |x - x_0| < 1$, we can write this as

$$s \leqslant \frac{\log |f(x) - f * \varphi_h(x)|}{\log(|h| + |x - x_0|)} - \frac{\log C_\epsilon}{\log(|h| + |x - x_0|)} + \epsilon.$$

Since the second term tends to zero when $|h| + |x - x_0| \to 0$, the condition is equivalent to

$$s \leqslant \frac{\log |f(x) - f * \varphi_h(x)|}{\log(|h| + |x - x_0|)} + \epsilon \qquad \text{for } |h| + |x - x_0| < \eta(\epsilon),$$

where $\epsilon > 0$ is arbitrarily small. By the maximality of $\alpha(x_0)$ and the very definition of lim inf, this implies that

$$\alpha(x_0) = \liminf_{h+|x-x_0|\to 0} \frac{\log|f(x) - f * \varphi_h(x)|}{\log(|h| + |x - x_0|)}.$$

If we define the functions α_j by

$$\alpha_j(x_0) = \inf_{2^{-j}\leqslant h+|x-x_0|<2\cdot 2^{-j}} \frac{\log(|f(x) - f * \varphi_h(x)| + 2^{-j^2})}{\log(|h| + |x - x_0|)},$$

then α_j are clearly continuous (since $f * \varphi_h(x)$ is continuous in (x, h), by dominated convergence), and $\alpha(x) = \liminf_{j\to\infty}\alpha_j(x)$. ■

For the sake of completeness we prove the converse as well by giving a construction of a continuous function f having Hölder exponent $\alpha_f(x) = \alpha(x)$, where $\alpha = \liminf \alpha_j$ and α_j are continuous functions. The construction essentially follows Jaffard [7]. To simplify notation, we describe it in one dimension; the extension to the n-dimensional case is straightforward.

The idea is to prescribe the wavelet coefficients of the function, using the following theorem, cf. [4]. We let $\{2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$ be an orthogonal wavelet basis of $L^2(\mathbb{R})$ with $\psi \in \mathcal{S}$.

THEOREM 4. *Let f be a bounded function. If $f \in C^s(x_0)$ then its wavelet coefficients $\alpha_{j,k} = \langle f, 2^j\psi(2^j x - k)\rangle$ satisfy*

$$|\alpha_{j,k}| \leqslant C(2^{-j} + |x_0 - k2^{-j}|)^s. \tag{4.1}$$

Conversely, if (4.1) *holds and $|\alpha_{j,k}| \leqslant C_m j^{-m}$ for all integers $m \in \mathbb{N}$ and $j \geqslant 1$, then $f \in C^{s-\epsilon}(x_0)$ for all $\epsilon > 0$.*

Proof of the converse part of Theorem 1. In view of Theorem 4, a natural attempt would be to set $f = \sum_{j\geqslant j_0}\sum_{k\in\mathbb{Z}} \alpha_{j,k}\psi_{j,k}$, with

$$\alpha_{j,k} = \min(2^{-j\alpha_j(k2^{-j})}, 2^{-j/\log j}).$$

This recipe actually works if, for instance, $\omega(\alpha_j, 2^{-j/(\log j)^2}) \to 0$ when $j \to \infty$, where $\omega(g, h) = \sup_{|x-y|\leqslant h}|g(x) - g(y)|$ denotes the uniform modulus of continuity. (Since the problem to be solved is local we may assume that each α_j is uniformly continuous so that $\omega(\alpha_j, h)$ exists and tends to 0 as $h \to 0$.) Before proving this, let us show that we can easily obtain this situation by ''slowing down'' the sequence α_j. More specifically we choose an increasing sequence of integers n_j so that $\omega(\alpha_j, 2^{-n_j/(\log n_j)^2}) \to 0$ when $j \to \infty$. Then we replace $\{\alpha_j\}$ by a new sequence $\{\tilde{\alpha}_n\}$, where we let $\tilde{\alpha}_{n_j} = \alpha_j$. For the remaining n we set $\tilde{\alpha}_n = \infty$, which amounts to simply setting $\alpha_{n,k} = 0$, so these n need not be considered. Clearly $\liminf \tilde{\alpha}_j = \liminf \alpha_j$, and we have $\omega(\tilde{\alpha}_j, 2^{-j/(\log j)^2}) \to 0$ as $j \to \infty$.

To prove that $\alpha_f(x) = \alpha(x)$ if $\omega(\alpha_j, 2^{-j/(\log j)^2}) \to 0$ when $j \to \infty$, we first note that, by the first part of Theorem 4,

$$\begin{aligned}\alpha_f(x) &\leqslant \liminf_{|x-k2^{-j}|\leqslant 2^{-j}} \alpha_j(k2^{-j}) \\ &\leqslant \liminf \alpha_j(x) + \liminf (\alpha_j(k2^{-j}) - \alpha_j(x)) = \alpha(x).\end{aligned}$$

To prove the reverse inequality we write

$$|\alpha_{j,k}| = \min(2^{-j\alpha(x)} 2^{j(\alpha(x)-\alpha_j(x))} 2^{j(\alpha_j(x)-\alpha_j(k2^{-j}))}, 2^{-j/\log j}). \tag{4.2}$$

For arbitrary $\epsilon > 0$, we have $\alpha(x) - \alpha_j(x) < \epsilon$ whenever j is sufficiently large, since $\alpha = \liminf \alpha_j$. Furthermore, if $|x - k2^{-j}| \leqslant 2^{-j/(\log j)^2}$ then $|\alpha_j(x) - \alpha_j(k2^{-j})| \leqslant \omega(\alpha_j, 2^{-j/(\log j)^2}) < \epsilon$ for j large enough, whereas for $|x - k2^{-j}| \geqslant 2^{-j/(\log j)^2}$, we have $2^{-j/\log j} \leqslant |x - k2^{-j}|^{\alpha(x)}$ for large j. Summing up, we find that

$$|\alpha_{j,k}| \leqslant 2^{-j(\alpha(x)-2\epsilon)} + |x - k2^{-j}|^{\alpha(x)}$$

for large j. The second part of Theorem 4 now tells us that $f \in C^{\alpha(x)-3\epsilon}(x)$, so $\alpha_f(x) \geqslant \alpha(x)$, and we conclude that $\alpha_f(x) = \alpha(x)$. ■

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